

# Some Reductive Methods for Nonlinear Waves in an Elastic Cylinder and a Channel Containing a Two-Layer Fluid

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A nonlinear wave theory is developed to review the reductive Taniuti-Wei, the derivative expansion, and the rays methods. Model equations describing wave propagation in a fluid-filled cylinder with a thin wall of elastic rings and a channel containing a two-layer fluid are derived via these methods.

## 1. Introduction

The study of pressure waves in tubes has a long history. Elastic tubes containing a fluid have been considered for the first time by Thomas Young [1] in connection with modelling the propagation of the arterial pressure. Moodie et al. [2] have recently discussed this problem assuming the fluid to be inviscid. A one-dimensional theory has been extracted by averaging the quantities over the tube cross section. The linear tube has been modelled on the basis of a viscoelastic shell theory which neglects bending moments and shear deformation of the wall. The problem was solved in the form of a Laplace integral which was inverted numerically. The results were tested against experimentally generated pressure pulses. The linear model has been developed to describe the motion and reflection of pressure waves in joined, initially uniform viscoelastic tubes and their subsequent interaction with various junctions characteristic of the arterial system [3]. Wave propagation and shock formation in elastic and viscoelastic fluid-filled tubes for a Mooney-Rivlin material have been discussed by Tait et al. [4]. Also the radial motion of a viscoelastic and thin-walled tube has been introduced by considering a model exhibiting impact and long time relaxed elastic behaviour [5]. A two-dimensional analysis was employed to study linear pulse propagation in thin-walled circularly cylindrical elastic tubes containing an inviscid and in-

compressible liquid. The results showed that the axial velocity can vary as much as 15 to 20 percent across a cross section [6]. A viscoelastic shell model for transient pressure perturbations in linear fluid-filled tubes has been presented and tested against experiments involving a water-filled latex tube. The experimental and numerical results are in good agreement over the major part of the wave [7].

In 1980 Lamb [8] showed, using Taniuti-Wei method [9], that in the absence of energy dissipation the fundamental set of nonlinear equations describing the irrotational motion of pressure waves in an incompressible liquid that is confined within an infinitely long circular cylinder with thin walls of elastic rings leads to the Korteweg-de Vries equation. For this case the same equation has been derived via Whitham's method [10] by Murawski [11]. On the contrary, when dissipation is taken into consideration the Burgers equation has been obtained [12].

Waves in a two-layer fluid have been considered by Kakutani et al. [13] and Funakoshi [14] as a simplest model of a stratified shear flow [15]. There may exist two modes which differ in the magnitude of their phase velocities [16]. For the fast mode, the surface and interface waves are in phase, and the amplitude of the surface wave is always larger than that of the interface one. On the other hand, for the slow mode these waves are always 180° out of phase [13]. We develop here a method presented by Shen and Zhong [17] in connection with water waves in a channel containing one layer of fluid.

The paper is arranged as follows: In the next section the fundamental set of equations describing

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waves in an elastic cylinder is presented. Sections 3 and 4 deal with the derivation of the Korteweg-de Vries and the nonlinear Schrödinger equations via the reductive Taniuti-Wei [9, 18] and the derivative expansion [19] methods. Section 5 presents the fundamental set of equations for two-fluid waves in a channel. The Korteweg-de Vries equation with varying coefficients is obtained in Sect. 6 via Shen's method [17]. For a critical density ratio (the nonlinear coefficient is equal to zero) the modified Korteweg-de Vries equation is derived in the same section. Near the critical density ratio the above equations are modified to combine them. Phase diagrams for this equation are presented in the Appendix. Section 7 is devoted to the concluding remarks.

## 2. Equations of Motion for Nonlinear Waves in a Cylinder

In Sects. 2–4 all investigations deal with one-dimensional irrotational incompressible fluid waves of characteristic amplitude  $l$  and characteristic length  $\lambda$  in an infinitely long tube with thin walls of elastic rings (undisturbed diameter  $2a$ ) to take into account dissipation of energy, nonlinearity and dispersion of the medium on the assumption that  $l \ll 2a \ll \lambda$ . Axial motions of the wall and bending moments are ignored. Then the fundamental set of equations may be written as

$$A_t + (AV)_x = 0, \quad (2.1)$$

$$V_t + VV_x + \frac{1}{\varrho} p_x = 0, \quad (2.2)$$

$$A_{tt} + \frac{E}{a^2 \varrho_m} A - \frac{2\pi a}{\varrho_m h} p = \frac{\pi(Eh - 2aq)}{\varrho_m h}, \quad (2.3)$$

where  $\varrho$  = constant liquid density,  $A$  = area of the cross section,  $V$  = liquid velocity,  $a$  = tube radius at the undisturbed uniform state,  $\varrho_m$  = density of the tube material,  $E$  = Young's modulus in the circumferential direction,  $p$  = liquid pressure,  $q$  = outside pressure,  $h$  = thickness of a tube wall. The subscripts  $x$  and  $t$  denote partial differentiation.

The model equation governing the motion of the tube wall as a linear viscoelastic solid characterized by its relaxation time was that employed in [20], but for our purpose (2.3) is used and may describe

impulse propagation in an arterial or nervous system. The experimental results obtained by Greenwald et al. [21] showed close agreement between the synthesised and measured waveforms. A problem of impulse propagation in nervous system was treated in [22].

By introducing the new coordinates

$$A \rightarrow \frac{1}{\pi a^2} A, \quad p \rightarrow \frac{2a}{Eh} p, \quad V \rightarrow \sqrt{\frac{2a\varrho}{Eh}} V, \\ t \rightarrow \frac{1}{a} \sqrt{\frac{E}{\varrho_m}} t, \quad x \rightarrow \frac{2\varrho}{\varrho_m a h} x, \quad (2.4)$$

dimensionless equations may be obtained, i.e.,

$$A_t + (AV)_x = 0, \quad (2.5)$$

$$V_t + VV_x + A_x + A_{xxt} = 0. \quad (2.6)$$

We define two dimensionless small parameters, namely

$$\varepsilon = \frac{2a}{\lambda}, \quad \delta = \frac{l}{2a}. \quad (2.7)$$

$\varepsilon$  and  $\delta$  measure the weakness of dispersion and nonlinearity, respectively. Various model equations may be derived depending on the relative magnitudes of these parameters. For example, the Korteweg-de Vries equation is obtained on the assumption that  $\delta = \varepsilon^2$  and the nonlinear Schrödinger equation corresponds to  $\delta = \varepsilon$ .

## 3. The Derivative Expansion Method for the Korteweg-de Vries Equation

As is mentioned in the Introduction, the fundamental set of equations (2.1)–(2.3) has been reduced for the first time to the Korteweg-de Vries equation by Lamb [8] using the reductive method. Also, the Lagrangian for the set of equations (2.5) and (2.6) was found and written as [11]

$$L = \frac{1}{2} A \psi_x^2 + A \psi_t - \frac{1}{2} A_t^2 + \frac{1}{2} A^2 - A + \frac{1}{2}, \quad V \equiv \psi_x. \quad (3.1)$$

The Euler-Lagrange equations lead to the Korteweg-de Vries equation.

Our purpose now is to show that a derivative expansion method leads to the same equation. We deal with small amplitudes and long waves. We

introduce the multiple spatial and temporal scales

$$t_n = \varepsilon^n t, \quad x_n = \varepsilon^n x, \quad n = 1, 2, \dots, \quad (3.2)$$

expand the dependent variables  $V$  and  $A$  around the undisturbed uniform state,

$$V = \delta V_1 + \delta^2 V_2 + \dots, \quad A = 1 + \delta A_1 + \delta^2 A_2 + \dots, \quad (3.3)$$

and assume that  $\delta = \varepsilon^2$ . On substituting (3.2) and (3.3) into (2.5) and (2.6) we get a sequence of equations by equating the coefficients of like powers of  $\varepsilon$ . From the equations for  $\varepsilon^2$  and  $\varepsilon^3$  we find

$$V_1(x_1, x_2, t_1, t_2) = V_1(\xi_1 \equiv x_1 - t_1, \xi_2 \equiv x_2 - t_2), \\ A_1 = V_1. \quad (3.4)$$

The fourth-order equations lead to

$$V_{2x_1x_1} - V_{2t_1t_1} + 2A_{1t_3\xi_1} + 2A_{1x_3\xi_1} + 3(A_1A_{1\xi_1})_{\xi_1} \\ + A_{1\xi_1\xi_1\xi_1} = 0. \quad (3.5)$$

The second-order terms may be removed if we assume that  $V_2$  depends on  $x_1$  and  $t_1$  through  $\xi_1$ . By transforming to a coordinate system moving with the phase velocity 1, i.e.,

$$\xi_3 = x_3 - t_3, \quad \tau = t_3, \quad (3.6)$$

we obtain the Korteweg-de Vries equation

$$A_{1\tau} + \frac{3}{2}A_1A_{1\xi_1} + \frac{1}{2}A_{1\xi_1\xi_1\xi_1} = 0. \quad (3.7)$$

This equation was modified to contain an integral term including the dissipative effects of viscous boundary layers in the context of water waves [23].

The basic Korteweg-de Vries equation and its solutions were discussed in some detail to obtain  $N$ -soliton [24] and  $N$ -periodical wave [25] solutions. Two-soliton scattering was considered in order to show the shape of the single soliton [26]. The stability of the Korteweg-de Vries waves and solitons was discussed by Infeld [27]. The Korteweg-de Vries equations for water waves were reviewed by Johnson [28].

#### 4. The Reductive Taniuti-Wei method for the Nonlinear Schrödinger Equation

In the previous section, we have applied the derivative expansion method to derive the Korte-

weg-de Vries equation for weakly dispersive waves. Here, turning our interest to strongly dispersive waves, we develop the reductive Taniuti-Wei method to obtain the nonlinear Schrödinger equation. For this aim we expand the quantities  $A$  and  $V$  into the series

$$A = 1 + \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-\infty}^{\infty} A_m^{(n)}(\xi, \tau) e^{im(kx - \omega t)}, \quad (4.1)$$

$$V = \sum_{n=1}^{\infty} \varepsilon^n \sum_{m=-\infty}^{\infty} V_m^{(n)}(\xi, \tau) e^{im(kx - \omega t)},$$

where

$$A_m^{(1)} = V_m^{(1)} = 0, \quad m \neq \pm 1, \quad A_m^{(1)*} = A_{-m}^{(1)}, \quad (4.2)$$

$$\xi = \varepsilon(x - \lambda t), \quad \tau = \varepsilon^2 t. \quad (4.3)$$

Substitution of the expansion (4.1) into (2.5) and (2.6) yields for the first-order terms

$$V_1^{(1)} = \frac{\omega}{k} A_1^{(1)}, \quad \omega^2 = \frac{k^2}{1 + k^2}. \quad (4.4), (4.5)$$

From the second-order equations we find

$$i(k V_1^{(2)} - \omega A_1^{(2)}) = \left( \lambda - \frac{\omega}{k} \right) A_{1\xi}^{(1)}, \quad (4.6)$$

$$A_2^{(2)} = \frac{1}{2k^2} (A_1^{(1)})^2, \quad (4.7)$$

$$V_2^{(2)} = \frac{(1 - 2k^2)\omega}{2k^3} (A_1^{(1)})^2, \quad (4.8)$$

$$\lambda = \omega_k. \quad (4.9)$$

The compatibility condition for the components of  $V_1^{(3)}$  and  $A_1^{(3)}$  can be reduced to the nonlinear Schrödinger equation

$$i A_{1\tau}^{(1)} + \frac{\omega^3(12k^6 + 35k^4 + 39k^2 + 9)}{4k^4(k^4 + 3k^2 + 3)} |A_1^{(1)}|^2 A_1^{(1)} \\ - \frac{3\omega}{2(1 + k^2)^2} A_{1\xi\xi}^{(1)} = C A_1^{(1)}, \quad (4.10)$$

where  $C$  is an integration constant which may be calculated from the boundary conditions.

It is well known that this equation may be solved by means of the inverse scattering transform [16], the direct method [29] and Wronskian determinants [30], and possesses Painlevé test and the Bäcklund transformation [31] to obtain an  $N$ -envelope soliton

solution and a periodic envelope one [32]. The stability of this equation was considered by Infeld and Rowlands [33].

### 5. The Fundamental Set of Equations for Water Waves in a Channel Containing a Two-Layer Fluid

We consider the motion of an inviscid, incompressible two-layer fluid of constant densities under gravity in an open channel with the bottom varying slowly in the longitudinal direction (Figure 1). We assume that the transverse velocities are much smaller than the longitudinal ones and develop the method presented by Shen and Zhong [17]. Our starting point is the following set of fundamental equations:

Upper layer:

$$u_{1x} + v_{1y} + w_{1z} = 0, \quad (5.1)$$

$$u_{1t} + u_1 u_{1x} + v_1 u_{1y} + w_1 u_{1z} + \frac{1}{\rho_1} p_{1x} = 0, \quad (5.2)$$

$$v_{1t} + u_1 v_{1x} + v_1 v_{1y} + w_1 v_{1z} + \frac{1}{\rho_1} p_{1y} = 0, \quad (5.3)$$

$$w_{1t} + u_1 w_{1x} + v_1 w_{1y} + w_1 w_{1z} + \frac{1}{\rho_1} p_{1z} + g = 0, \quad (5.4)$$

subject to the boundary conditions

$$\eta_{1t} + u_1 \eta_{1x} + v_1 \eta_{1y} = w_1, \quad (5.5)$$

$$p_1 = 0, \quad \text{at } z = \eta_1, \quad (5.6)$$

$$\eta_{2t} + u_1 (\eta_{2x} - h_{0,x}) + v_1 \eta_{2y} = w_1, \quad \text{at } z = \eta_2 - h_0, \quad (5.7)$$

$$u_1 D_x + v_1 D_y + w_1 D_z = 0, \quad \text{at } D = 0; \quad (5.8)$$

lower layer:

$$u_{2x} + v_{2y} + w_{2z} = 0, \quad (5.9)$$

$$u_{2t} + u_2 u_{2x} + v_2 u_{2y} + w_2 u_{2z} + \frac{1}{\rho_2} p_{2x} = 0, \quad (5.10)$$

$$v_{2t} + u_2 v_{2x} + v_2 v_{2y} + w_2 v_{2z} + \frac{1}{\rho_2} p_{2y} = 0, \quad (5.11)$$

$$w_{2t} + u_2 w_{2x} + v_2 w_{2y} + w_2 w_{2z} + \frac{1}{\rho_2} p_{2z} + g = 0, \quad (5.12)$$

subject to the boundary conditions

$$\eta_{2t} + u_2 (\eta_{2x} - h_{0,x}) + v_2 \eta_{2y} = w_2, \quad (5.13)$$

$$p_2 = p_1, \quad \text{at } z = \eta_2 - h_0, \quad (5.14)$$

$$u_2 D_x + v_2 D_y + w_2 D_z = 0, \quad \text{at } D = 0, \quad (5.15)$$

where  $(u_i, v_i, w_i)$ ,  $i = 1, 2$ , are the velocities,  $g$  is the gravitational acceleration,  $\rho_i$  are the constant densities,  $p_i$  are the pressures, and  $t$  is the time.

### 6. Shen's Method

#### a) Derivation of the Korteweg-de Vries Equation

We define here the large parameter

$$\sigma^{3/2} = \frac{L}{H} \gg 1, \quad (6.1a)$$

where  $L$  and  $H$  are respectively the longitudinal and transverse length scales. We further introduce the following dimensionless variables:

$$\left( \frac{g}{H} \right)^{1/2} \sigma^{-3/2} t \rightarrow t, \quad \left( \frac{\sigma^{-3/2} x}{H}, \frac{y}{H}, \frac{z}{H} \right) \rightarrow (x, y, z),$$

$$A = \frac{\rho_1}{\rho_2}, \quad \frac{\eta_i}{H} \rightarrow \eta_i, \quad \frac{D}{H} \rightarrow D,$$

$$\frac{p_i}{\rho_1 g H} \rightarrow p_i, \quad \frac{h_0}{H} \rightarrow h_0, \quad (6.2a)$$

$$\left( \frac{u_i}{(gH)^{1/2}}, \frac{\sigma^{1/2} v_i}{(gH)^{1/2}}, \frac{\sigma^{1/2} w_i}{(gH)^{1/2}} \right) \rightarrow (u_i, v_i, w_i).$$

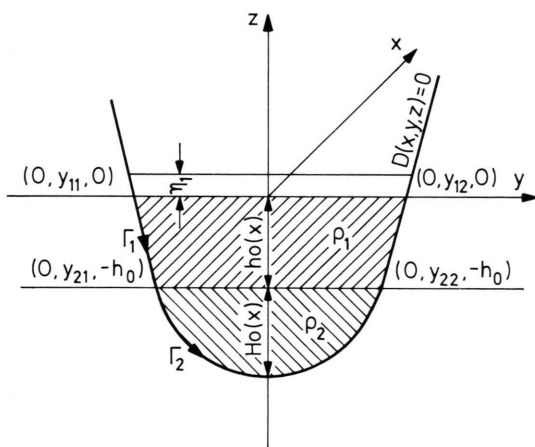


Fig. 1. A cross section of the channel.

We assume that  $u_i$ ,  $v_i$ ,  $w_i$ ,  $p_i$  and  $\eta_i$ , as functions of  $x$ ,  $y$ ,  $z$  and  $t$ , depend explicitly on the new variable

$$\zeta = \sigma \zeta(x, t), \quad (6.3a)$$

where  $\zeta$  will be called a phase function, and that they possess an asymptotic expansion of the form

$$\psi_i = \psi_i^{(0)} + \sigma^{-1} \psi_i^{(1)} + \dots, \quad (6.4a)$$

where  $\psi_i^{(0)}$  is a vector of the undisturbed uniform state:

$$(u_i^{(0)}, v_i^{(0)}, w_i^{(0)}) = 0, \quad p_1^{(0)} = -z, \quad (6.5a)$$

$$p_2^{(0)} = h_0 - \frac{1}{\Delta}(z + h_0), \quad \eta_i^{(0)} = 0.$$

Substitution of (6.2a)–(6.5a) into (5.1)–(5.15) and use of the expressions

$$\psi(z = \eta_1) = \psi(z = 0) + \psi_z \eta_1 + O(\eta_1^2), \quad (6.6a)$$

$$\psi(z = \eta_2 - h_0) = \psi(z = -h_0) + \psi_z \eta_2 + O(\eta_2^2)$$

will yield a sequence of equations since the coefficients of like powers of  $\sigma^{-1}$  must be zero. The first-order equations are for the upper layer

$$k u_{1\zeta}^{(1)} + v_{1y}^{(1)} + w_{1z}^{(1)} = 0, \quad (6.7a)$$

$$k p_{1\zeta}^{(1)} = \Omega u_{1\zeta}^{(1)}, \quad (6.8a)$$

$$p_{1y}^{(1)} = p_{1z}^{(1)} = 0, \quad (6.9a)$$

$$\left. \begin{aligned} w_1^{(1)} + \Omega \eta_{1\zeta}^{(1)} &= 0, \\ p_1^{(1)} &= \eta_1^{(1)}, \end{aligned} \right\} \quad \text{at } z = 0, \quad (6.10a)$$

$$w_1^{(1)} + \Omega \eta_{2\zeta}^{(1)} = 0, \quad \text{at } z = -h_0, \quad (6.12a)$$

$$v_1^{(1)} D_y + w_1^{(1)} D_z = 0, \quad \text{at } D = 0, \quad (6.13a)$$

and for the lower layer

$$k u_{2\zeta}^{(1)} + v_{2y}^{(1)} + w_{2z}^{(1)} = 0, \quad (6.14a)$$

$$\Delta k p_{2\zeta}^{(1)} = \Omega u_{2\zeta}^{(1)}, \quad (6.15a)$$

$$p_{2y}^{(1)} = p_{2z}^{(1)} = 0, \quad (6.16a)$$

$$\left. \begin{aligned} w_2^{(1)} + \Omega \eta_{2\zeta}^{(1)} &= 0, \\ \eta_2^{(1)} &= \frac{\Delta}{1 - \Delta} (p_2^{(1)} - p_1^{(1)}), \end{aligned} \right\} \quad \text{at } z = -h_0, \quad (6.18a)$$

$$v_2^{(1)} D_y + w_2^{(1)} D_z = 0, \quad \text{at } D = 0, \quad (6.19a)$$

where

$$k \equiv \zeta_x, \quad \Omega \equiv -\zeta_t. \quad (6.20a)$$

From (6.9a) and (6.16a), we may define

$$p_1^{(1)} \equiv A(x, t, \zeta), \quad p_2^{(1)} \equiv B(x, t, \zeta). \quad (6.21a)$$

Multiplying (6.7a) by  $\Omega$  and (6.8a) by  $k$ , adding and integrating over the cross section  $S_1$ , we obtain

$$\frac{k^2 S_1}{\Omega} A_\zeta = \frac{\Delta b \Omega}{\Delta - 1} (B_\zeta - A_\zeta) + a \Omega A_\zeta, \quad (6.22a)$$

where  $a$  and  $b$  are the widths of the channel in the planes  $z = 0$  and  $z = -h_0$ , respectively. By a similar procedure we get from (6.14a) and (6.15a)

$$A_\zeta = \frac{k^2 S_2 (\Delta - 1) + b \Omega^2}{b \Omega^2} B_\zeta \equiv \mathcal{H} B_\zeta. \quad (6.23a)$$

So, we find the linear dispersion relation

$$ab \Omega^4 - [a S_2 (1 - \Delta) + b (\Delta S_2 + S_1)] k^2 \Omega^2 + S_1 S_2 (1 - \Delta) k^4 = 0. \quad (6.24a)$$

Hence, it follows that two modes are possible. From the second-order equations we obtain

$$p_1^{(2)} = -\varphi_1(x, y, z, t) A_{\zeta\zeta}, \quad (6.25a)$$

$$p_2^{(2)} = -\varphi_2(x, y, z, t) B_{\zeta\zeta},$$

where the potentials  $\varphi_1$  and  $\varphi_2$  satisfy the following Neuman problems:

$$\varphi_{iyy} + \varphi_{izz} = k^2, \quad (6.26a)$$

$$\varphi_{iy} D_y + \varphi_{iz} D_z = 0, \quad \text{at } D = 0, \quad (6.27a)$$

$$\varphi_{1z} = \Omega^2, \quad \text{at } z = 0, \quad (6.28a)$$

$$\varphi_1 A_{\zeta\zeta} - \varphi_2 B_{\zeta\zeta} = \frac{1 - \Delta}{\Delta} \eta_2^{(2)}, \quad \text{at } z = -h_0. \quad (6.29a)$$

The compatibility condition is reduced to the Korteweg-de Vries equation for the first-order quantity  $B$ :

$$m_1 B_t + m_2 B_x + m_3 B B_\zeta + m_4 B_{\zeta\zeta\zeta} + m_5 B = 0, \quad (6.30a)$$

where

$$m_1 = a \Omega \mathcal{H} - \frac{\Delta b (1 - \mathcal{H}) \Omega}{1 - \Delta} + \frac{k^2 S_1 \mathcal{H}}{\Omega}, \quad (6.31a)$$

$$m_2 = 2k S_1 \mathcal{H}, \quad (6.32a)$$

$$m_3 = \frac{\Delta b \mathcal{H} (1 - \mathcal{H}) k^2}{\Delta - 1} - \frac{\Delta (1 - \mathcal{H}) \mathcal{H}}{1 - \Delta} \int_{y_{21}}^{y_{22}} \varphi_{1zz}|_{z=-h_0} dl \quad (6.33a)$$

$$+ \mathcal{H}^2 \int_{y_{11}}^{y_{12}} (\varphi_{1zz}|_{z=0} + k^2) dl + \frac{k^4 S_1 \mathcal{H}^2}{\Omega^2},$$

$$m_4 = \frac{\Delta \Omega^2}{1 - \Delta} \left\{ \mathcal{H} \int_{y_{21}}^{y_{22}} \varphi_1|_{z=-h_0} dl - \int_{y_{21}}^{y_{22}} \varphi_2|_{z=-h_0} dl \right\}$$

$$+ \Omega^2 \mathcal{H} \int_{y_{11}}^{y_{12}} \varphi_1|_{z=0} dl - k^2 \mathcal{H} \iint_{S_1} \varphi_1 dS_1, \quad (6.34a)$$

$$m_5 = -k \mathcal{H} \int_{\Gamma_1} \frac{D_x dl}{(D_y^2 + D_z^2)^{1/2}} + k b h_{0x} \mathcal{H} + 2k S_1 \mathcal{H}_x. \quad (6.35a)$$

Equation (6.24a) may be solved by means of the method of characteristics determined by

$$\frac{dx}{dt} = \frac{\Omega}{k}, \quad \frac{dk}{dt} = -k \left( \frac{\Omega}{k} \right)_x, \quad \frac{d\Omega}{dt} = \frac{d\zeta}{dt} = 0. \quad (6.36a)$$

Along a ray

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\Omega}{k} \frac{\partial}{\partial x}, \quad (6.37a)$$

and (6.30a) may be rewritten as

$$m_1 B_t + m_3 B B_\xi + m_4 B_{\xi\xi\xi} + m_5 B = 0. \quad (6.38a)$$

A solution of this equation was considered in the context of a solitary wave propagation from one uniform cross section of a symmetric triangular channel into another through a transition region to find the criteria for the fission of solitons. The numerical results showed that the solitary wave is desintegrated into a train of solitons of decreasing amplitudes [17], in qualitative agreement with sea waves. The Korteweg-de Vries equation was also derived by Johnson [34] for a channel of variable depth and appears in various physical systems [35].

#### b) Derivation of the Modified Korteweg-de Vries Equations

The nonlinear coefficient  $m_3$  may become zero for some critical value of  $\Delta = \Delta^*$ . There a stronger nonlinear effect should be taken into consideration

to adopt the expansion

$$\psi_i = \psi_i^{(0)} + \sigma^{-1/2} \psi_i^{(1)} + \sigma^{-1} \psi_i^{(2)} + \dots, \quad (6.1b)$$

where  $\psi_i^{(0)}$  is the same undisturbed uniform state vector. The equations for the second approximation ( $\sigma^{-1}$ ) are for the upper layer

$$k u_{1\xi}^{(2)} + v_{1y}^{(2)} + w_{1z}^{(2)} = 0, \quad (6.2b)$$

$$k p_{1\xi}^{(2)} + k u_1^{(1)} u_{1\xi}^{(1)} = \Omega u_{1\xi}^{(2)}, \quad (6.3b)$$

$$p_{1y}^{(2)} = p_{1z}^{(2)} = 0, \quad (6.4b)$$

$$k u_1^{(1)} \eta_{1\xi}^{(1)} = \Omega \eta_{1\xi}^{(2)} + w_1^{(2)} + w_{1z}^{(1)} \eta_1^{(1)}, \quad (6.5b)$$

$$p_1^{(2)} = \eta_1^{(2)}, \quad \text{at } z = 0, \quad (6.6b)$$

$$v_1^{(2)} D_y + w_1^{(2)} D_z = 0, \quad \text{at } D = 0, \quad (6.7b)$$

$$k u_1^{(1)} \eta_{2\xi}^{(1)} + v_1^{(1)} \eta_{2y}^{(1)} = \Omega \eta_{2\xi}^{(2)} + w_1^{(2)} + w_{1z}^{(1)} \eta_2^{(1)}, \quad \text{at } z = -h_0, \quad (6.8b)$$

and for the lower layer

$$k u_{2\xi}^{(2)} + v_{2y}^{(2)} + w_{2z}^{(2)} = 0, \quad (6.9b)$$

$$k \Delta^* p_{2\xi}^{(2)} + k u_2^{(1)} u_{2\xi}^{(1)} = \Omega u_{2\xi}^{(2)}, \quad (6.10b)$$

$$p_{2y}^{(2)} = p_{2z}^{(2)} = 0, \quad (6.11b)$$

$$k u_2^{(1)} \eta_{2\xi}^{(1)} + v_2^{(1)} \eta_{2y}^{(1)} = \Omega \eta_{2\xi}^{(2)} + w_2^{(2)} + w_{2z}^{(1)} \eta_2^{(1)}, \quad \text{at } z = -h_0, \quad (6.12b)$$

$$\eta_2^{(2)} = \frac{\Delta^*}{\Delta^* - 1} (p_1^{(2)} - p_2^{(2)}), \quad \text{at } z = -h_0, \quad (6.13b)$$

$$v_2^{(2)} D_y + w_2^{(2)} D_z = 0, \quad \text{at } D = 0. \quad (6.14b)$$

Applying the divergence theorem, from  $\sigma^{-3/2}$  we obtain along a ray the modified Korteweg-de Vries equation

$$n_1 B_t + n_3 B^2 B_\xi + n_4 B_{\xi\xi\xi} + n_5 B = 0, \quad (6.15b)$$

where

$$n_1 = a \Omega \mathcal{H} - \frac{\Delta^* (1 - \mathcal{H}) b \Omega}{1 - \Delta^*} + \frac{k^2 \mathcal{H} S_1}{\Omega}, \quad (6.16b)$$



$$n_3 = 2ak^2 \mathcal{H} + ak \Omega \tilde{\mathcal{I}} \mathcal{H} + \mathcal{H} \mathcal{H} \int_{\gamma_{11}}^{\gamma_{12}} \varphi_{1zz}|_{z=0} dl + \frac{\mathcal{H}(\mathcal{H} - \mathcal{I})}{1 - \Delta^*} \int_{\gamma_{21}}^{\gamma_{22}} \varphi_{1zz}|_{z=-h_0} dl \\ + \frac{-k \Delta^* \Omega \tilde{\mathcal{I}} b(1 - \mathcal{H})}{1 - \Delta^*} + \frac{2\Delta^* k^2 \mathcal{H}(\mathcal{H} - \tilde{\mathcal{I}}) b}{1 - \Delta^*} + \frac{3k^3 \tilde{\mathcal{I}} \mathcal{H} S_1}{\Omega}, \quad (6.17b)$$

$$n_4 = \Omega^2 \mathcal{H} \int_{\gamma_{11}}^{\gamma_{12}} \varphi_1|_{z=0} dl + \frac{\Delta^* \Omega^2}{\Delta^* - 1} \int_{\gamma_{21}}^{\gamma_{22}} (\varphi_2 - \mathcal{H} \varphi_1)|_{z=-h_0} dl - k^2 \mathcal{H} \iint_{S_1} \varphi_1 dS_1, \quad (6.18b)$$

$$n_5 = 2k S_1 \mathcal{H}_x - k \mathcal{H} \int_{\Gamma_1} \frac{D_x dl}{(D_y^2 + D_z^2)^{1/2}}, \quad (6.19b)$$

$$\mathcal{I} = C_1 + C_2 \mathcal{H}, \quad \tilde{\mathcal{I}} = \frac{2\mathcal{H} \Omega^2 + k^3 \mathcal{H}^2}{2\Omega^3}, \quad (6.20b)$$

$$\mathcal{H} = \frac{\Delta^* k^2 b \Omega^2 (1 - \mathcal{H}) + \Delta^* k^4 S_2 (1 - \Delta^*) + 2C_1 \Omega^2 [\Delta^* (1 - \Delta^*) k^2 S_2 - \Delta^* \Omega^2 b]}{2\Omega^2 \{C_2 [\Delta^* (\Delta^* - 1) k^2 S_2 + \Delta^* b \Omega^2] - \Delta^* b \Omega^2\}}, \quad (6.21b)$$

$$C_1 = \frac{a k^2 \Omega^2 \mathcal{H}^2 (\Delta^* - 1) + k^2 \Omega^2 \Delta^* b \mathcal{H} (1 - \mathcal{H}) + k^4 S_1 \mathcal{H}^2 (\Delta^* - 1)}{2b \Delta^* \Omega^4}, \quad (6.22b)$$

$$C_2 = \frac{k^2 S_1 (\Delta^* - 1) - \Delta^* b \Omega^2 - a (\Delta^* - 1) \Omega^2}{\Delta^* b \Omega^2}. \quad (6.23b)$$

$\varphi_1$  and  $\varphi_2$  satisfy (6.26 a) – (6.29 a) in which  $\eta_2^{(2)}$  is replaced by  $\eta_2^{(3)}$ .

Equation (6.15b) with constant coefficients and  $n_5 = 0$  was studied in some details to obtain one-soliton, brother-like and two-brother-like solutions [8]. Bäcklund transformations relating this equation and the Korteweg-de Vries equation were found [36]. The trace method [37] and the Fredholm determinant method [38] were used to construct the solutions. The representation of periodic waves as sums of solitons was given [39].

Near the critical density ratio we have to modify our previous procedure and assume that

$$\Delta - \Delta^* = \sigma^{-1/2}. \quad (6.24b)$$

From the third-order equations ( $\sigma^{-3/2}$ ), we get the mixed Korteweg-de Vries and modified Korteweg-de Vries equation of the form

$$n_1 B_t + m_3 B B_{\xi} + n_3 B^2 B_{\xi} + n_4 B_{\xi\xi\xi} + n_5 B = 0, \quad (6.25b)$$

where the coefficients  $n_1, n_3 - n_5, m_3$  coincide exactly with coefficients of equations (6.38a) and (6.15b). This equation for  $n_5 = 0$  and constant coefficients was derived by Kakutani et al. [13] and

Funakoshi [14]. They showed that this equation possesses a shock-like solution and a periodic one expressed in terms of Jacobian elliptic functions. The phase-diagrams are constructed in the Appendix to present the whole set of stationary solutions.

## 7. Summary

In the present paper we reviewed some reductive methods, Taniuti-Wei, reductive expansion, Whitham's, and rays to derive the Korteweg-de Vries, the modified Korteweg-de Vries, the mixed Korteweg-de Vries, modified Korteweg-de Vries, and the nonlinear Schrödinger equations. The Korteweg-de Vries equation for liquid waves in a cylinder was previously obtained via Taniuti-Wei [8] and the Whitham's [11] methods and agrees with (3.7), which is found via the derivative expansion method. It supports the convergence of all reductive methods.

It is worth noticing that the two coefficients of the second and third terms in (4.10) are responsible for modulational stability of waves [40]. If their product

is negative, nonlinear wave solutions are modulationally stable. In our case this occurs for arbitrary values of  $k$ .

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## Appendix

We rewrite (6.25 b) in the form

$$B_t + \alpha_1 B B_x + \alpha_2 B^2 B_x + \alpha_3 B_{xxx} = 0 \quad (\text{A.1})$$

and look for stationary solutions of the form

$$B = B(\zeta = x - ct). \quad (\text{A.2})$$

Upon integration of (A.1) one gets

$$\alpha_3 B_{\zeta\zeta} - cB + \frac{\alpha_1}{2} B^2 + \frac{\alpha_2}{3} B^3 = a, \quad (\text{A.3})$$

where  $a$  is an integration constant. Multiplication of (A.3) with  $B_{\zeta}$  yields

$$B_{\zeta}^2 = -\frac{\alpha_2}{6\alpha_3} B^4 - \frac{\alpha_1}{3\alpha_3} B^3 + \frac{c}{\alpha_3} B^2 + \frac{2a}{\alpha_3} B + b, \quad (\text{A.4})$$

$b = \text{const.}$

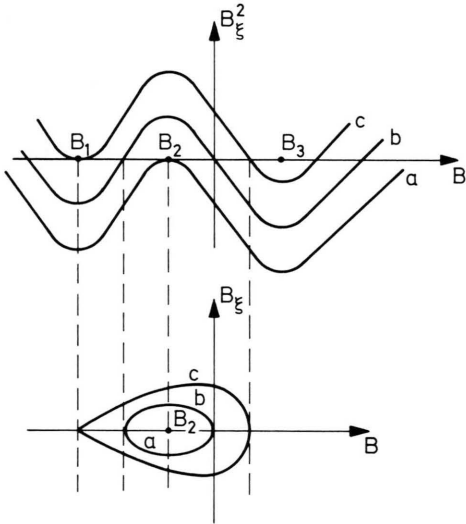


Fig. 2. Phase diagrams for  $\alpha_2 \alpha_3 < 0$ . a: Linear wave limit, b: conoidal wave, c: soliton.

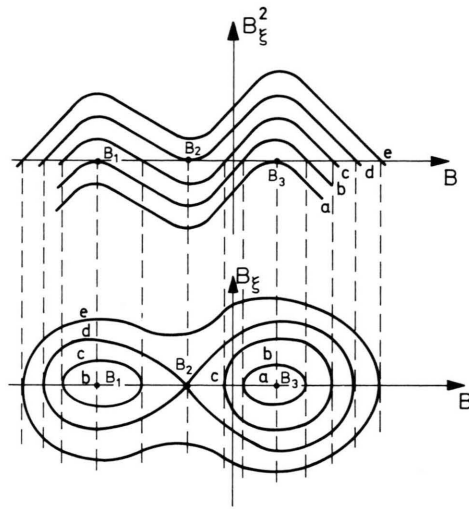


Fig. 3. Phase diagrams for  $\alpha_2 \alpha_3 > 0$ . a: Linear wave limit, b: linear wave limit on the left and conoidal wave on the right, c: conoidal waves, d: soliton, e: conoidal wave.

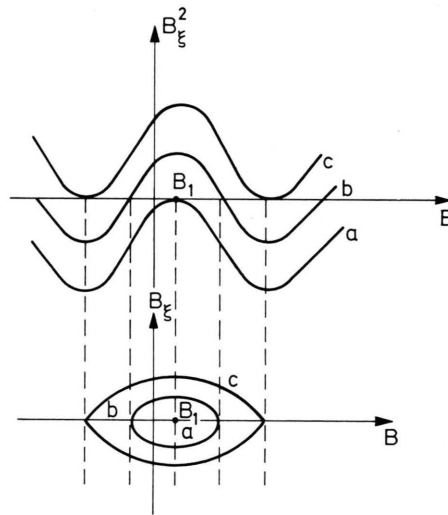


Fig. 4. Phase diagrams for  $\alpha_2 \alpha_3 > 0$  (special case). a: Linear wave limit, b: conoidal wave, c: shock wave.



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